Engineering Notes

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Optimum Propeller in Viscous Flow

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Nomenclature

b = blade width at radius r
C_D = sectional drag coefficient
C_L = sectional lift coefficient
E = C_D/C_L
N = power
R = propeller radius
r_h = radius of the hub
T = thrust
V = propeller translation
W_1 = resultant velocity
z = number of propeller blades
Γ = circulation around a blade at radius r
β_1 = aerodynamic pitch angle
ρ = density of fluid
ω = angular speed of propeller

Introduction

A DISCUSSION of propeller theory is not complete without a discussion of the optimum propeller. By the optimum propeller is meant the propeller where the aerodynamic-pitch distribution and corresponding circulation distribution produces maximum efficiency. Many efforts have been made and expressions for $\tan\beta_1$ are known for a free running propeller with an arbitrary number of blades in nonviscous flow. An expression is also known for an infinitly bladed propeller in viscous flow.

In this paper an expression for $\tan\beta_1$ is derived. The expression is valid for a propeller with an arbitrary number of blades, in viscous flow. In the deducing, Calculus of Variations is used, and the assumptions made are: Kutta-Joukowski's law must be valid and the blades must be symmetrically arranged. As an instance of where the derived expression may be used, designing of a moderatly loaded propeller may be mentioned. If the method given by Lerbs[1] is to be used, then the $\tan\beta_1$-distribution must be known.

Basic Theoretical Considerations

A reference system which rotates with the propeller is used, since the motion is steady with respect to this system.

When the above-mentioned assumptions are made, it is known from Ref. 2 that thrust and power may be written as

$$T = 1/2z\rho\int_{r_h}^{R} W_1^2 b\cdot(C_L \cos\beta_1 - C_D \sin\beta_1)dr \qquad (1)$$

$$N = 1/2z\rho\int_{r_h}^{R} W_1^2 b\cdot(C_L \sin\beta_1 + C_D \cos\beta_1)r\omega dr \qquad (2)$$

By introduction of the circulation and of following dimensionless variables, Eqs. (1) and (2) may be written on dimensionless form

$$x = r/R$$

$$G = \Gamma/(4\pi\omega R^2) = (1/2C_L W_1 b)/(4\pi\omega R^2)$$

$$v_1 = W_1 \sin\beta_1/\omega R$$

$$u_1 = W_1 \cos\beta_1/\omega R$$

$$T_c = T/(4\pi\rho\omega^2 R^4)$$

$$N_c = N/(4\pi\rho\omega^3 R^5)$$

$$T_c = z\int_{x_h}^{1} G(u_1 - Ev_1)\cdot dx \qquad (3)$$

$$N_c = z\int_{x_h}^{1} G(v_1 + Eu_1)x\cdot dx \qquad (4)$$

Finally, the efficiency is introduced

$$\eta = \lambda T_c/N_c \qquad (5)$$

where

$$\lambda = V/\omega R$$

Problem Formulation

Our objective is to find that particular function G which achieves prescribed values at the endpoints: $G(x_h) = k_1$ and $G(1) = k_2$ and maximize the efficiency when following parameters are kept constant

$$\rho, z, V, \omega, r_h, R \text{ and } E(x)$$

The problem is divided into two cases. In the first case it is assumed that the power is a known quantity, and in the second case, the thrust is assumed to be known.

Case of Known Power

When the power is a known quantity, then the function G we want to determine must achieve the required values at the endpoints and must maximize

$$I = \int_{x_h}^{1} G(u_1 - Ev_1)dx = \int_{x_h}^{1} G\cdot f\cdot dx \qquad (6)$$

while

$$J = \int_{x_h}^{1} G(v_1 + Eu_1)x\cdot dx = \int_{xh}^{1} G\cdot g\cdot dx = \ell = \text{constant} \qquad (7)$$

We assume that $\bar{G}$ is such a function and we now consider a varied function, which must be of a two-parameter family to be sure that the condition (7) is maintained

$$\phi = \bar{G} + \epsilon_1 G_1 + \epsilon_2 G_2 \qquad (8)$$

in which $G_1(x)$ and $G_2(x)$ are arbitrary functions for which

$$G_1(x_h) = G_1(1) = G_2(x_h) = G_2(1) = 0 \qquad (9)$$

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The condition (9) ensures that

$$\phi(x_h) = k_1 \text{ and } \phi(1) = k_2$$

as prescribed for all values of the parameters ϵ_1 and ϵ_1. We replace G by ϕ in Eq.'s (6) and (7)

$$I(\epsilon_1, \epsilon_2) = \int_{x_h}^{1} (\bar{G} + \epsilon_1 G_1 + \epsilon_2 G_2)\cdot f(\epsilon_1, \epsilon_2)\cdot dx \quad (10)$$

$$J(\epsilon_1, \epsilon_2) = \int_{x_h}^{1} (\bar{G} + \epsilon_1 G_1 + \epsilon_2 G_2)\cdot g(\epsilon_1, \epsilon_2)\cdot dx \quad (11)$$

Because of (11) it is seen that ϵ_1 and ϵ_2 are not independent. The relation between them is

$$J(\epsilon_1, \epsilon_2) = \ell \quad (12)$$

Our problem is then to maximize (10) while condition (12) is satisfied. To solve this problem we use the method of Lagrange multipliers described in Ref. 3. We introduce a new function of ϵ_1, ϵ_2

$$\psi(\epsilon_1, \epsilon_2) = I(\epsilon_1, \epsilon_2) + K\{\ell - J(\epsilon_1, \epsilon_2)\} \quad (13)$$

where K is a undetermined constant, the Lagrangian multiplier.

About ψ we know that for same value of (ϵ_1,ϵ_1), ψ gives the same result as I, and since by hypothesis affords a maximum to I, then we know

$$\frac{\partial\psi}{\partial\epsilon_1} = \frac{\partial\psi}{\partial\epsilon_2} = 0 \quad \text{when} \quad \epsilon_1 = \epsilon_2 = 0 \quad (14)$$

Now

$$\frac{\partial\psi}{\partial\epsilon_1}(\epsilon_1, \epsilon_2) = \int_{x_h}^{1} \{G_1\cdot f(\epsilon_1, \epsilon_2) + \phi(\epsilon_1, \epsilon_2)\cdot \frac{\partial f}{\partial\epsilon_1}(\epsilon_1, \epsilon_2)\}dx$$

$$- K\int_{x_h}^{1} \{G_1\cdot g(\epsilon_1, \epsilon_2) + \phi(\epsilon_1, \epsilon_2)\frac{\partial g}{\partial\epsilon_1}(\epsilon_1, \epsilon_2)\}dx \quad (15)$$

Setting $\epsilon_1 = \epsilon_2 = 0$ we have

$$\frac{\partial\psi}{\partial\epsilon_1}(0, 0) = \int_{x_h}^{1} G_1(f - Kg)dx$$

$$+ \int_{x_h}^{1} \bar{G}\frac{\partial}{\partial\epsilon_1}(f(\epsilon_1, \epsilon_2) - Kg(\epsilon_1, \epsilon_2))dx = 0$$

$$\text{when} \quad \epsilon_1 = \epsilon_2 = 0 \quad (16)$$

In the same way we have

$$\frac{\partial\psi}{\partial\epsilon_2}(0, 0) = \int_{x_h}^{1} G_2(f - Kg)dx$$

$$+ \int_{x_h}^{1} \bar{G}\frac{\partial}{\partial\epsilon_2}(f(\epsilon_1, \epsilon_2) - Kg(\epsilon_1, \epsilon_2))dx = 0$$

$$\text{when } \epsilon_1 = \epsilon_2 = 0 \quad (17)$$

Because of the arbitrary character of the functions G_1 and G_2 then the two relations (16) and (17) are essentially one. A solution for one of the equations will also be a solution for the other.

If, in Eq. (16), we assume that the first integral is equal to minus the second integral and different from zero then we restrain the arbitrary character of G_1, i.e.,

$$\int_{x_h}^{1} G_1(f - Kg)dx = 0 \quad \text{when} \quad \epsilon_1 = \epsilon_2 = 0 \quad (18)$$

and at same time

$$\int_{x_h}^{1} \bar{G}\frac{\partial}{\partial\epsilon_1}(f(\epsilon_1, \epsilon_2) - Kg(\epsilon_1, \epsilon_2))dx = 0$$

$$\text{when } \epsilon_1 = \epsilon_2 = 0 \quad (19)$$

We know from the basic lemma of Calculus of Variations,[3] that because of (9), (18) implies

$$f - Kg = 0 \quad (20)$$

If, in Eq. (20), we introduce the expressions of f and g, defined in Eqs. (6) and (7), we obtain

$$u_1 - Ev_1 = Kx(v_1 + Eu_1)$$

or since $\tan\beta_1 = v_1/u_1$

$$\tan\beta_1 = (1 - KEx)/(E + Kx) \quad (21)$$

It means that Eq. (21) is the only solution to Eq. (18), and furthermore we see

$$f(\epsilon_1, \epsilon_2) - Kg(\epsilon_1, \epsilon_2) =$$

$$u_1(\epsilon_1, \epsilon_2)\{E + Kx\}\cdot\{(1 - KEx)/(E + Kx) - \tan\beta_1\} \quad (22)$$

The relation (21) is therefore the only solution to (16) which does not restrain G_1, and that particular $\tan\beta_1$-distribution that implies maximum efficiency.

The Case of Known Thrust

In this case the function G we seek must achieve prescribed values at x_h and 1 and must minimize

$$J = \int_{x_h}^{1} G\cdot g\cdot dx \quad (23)$$

while

$$I = \int_{x_h}^{1} G\cdot f\cdot dx = \ell_1 = \text{constant} \quad (24)$$

We use the method described above and the equation equal to (13) becomes

$$\Theta(\epsilon_1, \epsilon_2) = J(\epsilon_1, \epsilon_2) + 1/K\{\ell_1 - I(\epsilon_1, \epsilon_2)\} \quad (25)$$

where $1/K$ is the Lagrangian multiplier, and it is seen that it leads us to the expression (21).

Conclusion and Discussion

We have by means of Calculus of Variations shown that in viscous flow

$$\tan\beta_1 = (1 - KEx)/(E + Kx)$$

implies maximum efficiency. The expression is derived on condition that Kutta-Joukowski's law is valid and further that the blades are symmetrically arranged. It is seen that when C_D tends to zero, we obtain the well known expression: $r\cdot\tan\beta_1$ = constant. It is also pointed out that the viscous drag/lift ratio E may vary along the radius, but the variation must be known from the outset, i.e., using of the derived expression in the designing procedure lead to a double iterative procedure, since also the value of K must be determined iteratively.

References

[1]Lerbs, H. W., "Moderately Loaded Propellers with a Finite Number of Blades and an Arbitrary Distribution of Circulation," Transactions of the Society of Naval Architects Marine Engineers, Vol. 60, 1952, pp. 73–123.

[2]Durand, W. F., "Aerodynamic Theory," Vol. IV, Division L, pp. 226–231.

[3]Weinstock, R., *Calculus of Variations*, McGraw-Hill, New York, 1952, pp. 17, 48–50.